On Solving Bivariate Unconstrained Optimization Problems Using Interval Analysis

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Abstract— In this paper, a method to find the optimal solution of bivariate unconstrained problems is proposed using Newton's interval analysis method. Interval Analysis method gives more accurate solution even for higher order derivatives. MATLAB programs are also developed for the procedure.

Index Terms— Interval Analysis, Newton's method, Unconstrained Optimization Techniques

1 Introduction

INTERVAL analysis method was evolved in 1950-1960s on the advent of Computational Mathematics. Interval analysis is a means of representing uncertainty by replacing a single (fixed point) value with an interval. Interval analysis is applied to numerical methods that deal with optimization functions. For a more detailed study, one can refer [1-6].

An attempt has been made to solve nonlinear unconstrained optimization problems on two variables. Based on [7], we have calculated the minimum of a bivariate function for linear derivatives that gives approximate values only. Using interval analysis method, calculating minima even for higher order derivatives is not tedious and it gives more accurate value than any other method.

The paper is organized as follows: Section 2 gives operation on interval arithmetic and section 3 deals with Newton's method. The extension of this method for unconstrained optimization problems with two variables is discussed in section 4. In section 5, a numerical example is provided to verify the feasibility of the proposed method. Finally some concluding remarks are given at the last.

2. Interval Arithmetic

The Interval Arithmetic Operations are given below as explained in [7].

Let $\tilde{x} = [x_1, x_2]$, $\tilde{y} = [y_1, y_2]$

(i) Addition:

$\tilde{x} + \tilde{y} = [x_1 + y_1, x_2 + y_2]$

(ii) Subtraction

$\tilde{x} - \tilde{y} = [x_1 - y_2, x_2 - y_1]$

(iii) Multiplication

$\tilde{x}.\tilde{y} = [\min(x_1y_1, x_1y_2, x_2y_1, x_2y_2), \max(x_1y_1, x_1y_2, x_2y_1, x_2y_2)]$

(iv) Division

$\frac{[a,b]}{[c,d]} = [a,b].\left[\frac{1}{d}, \frac{1}{c}\right]$ if $0 \notin [c,d]$

(v) $\lambda\tilde{x} = [\lambda x_1, \lambda x_2]$ for $\lambda \geq 0$

$= [\lambda x_2, \lambda x_1]$ for $\lambda < 0$

(vi) Inverse

$[x_1, x_2]^{-1} = \left[\frac{1}{x_2}, \frac{1}{x_1}\right]$, for $0 \notin [x_1, x_2]$

(vii) $[x_1, x_2]^n = [{x_1}^n, {x_2}^n]$, if $x_1 \geq 0$

$= [{x_2}^n, {x_1}^n]$, if $x_1 < 0$

$= [0, \max\{{x_1}^n, {x_2}^n\}]$, otherwise.

3. Newton's Method

Newton's method for multivariable optimization is analogues to Newton's single variable algorithm for obtaining the roots and Newton-Raphson method for finding the roots of first derivative for a given x_0, iterates

$$x_{k+1} = x_k - \left[\frac{f'(x_k)}{f''(x_k)}\right] \text{ until } |x_{k+1} - x_k| < \epsilon \tag{1}$$

Now, for the nonlinear equation of the type f(x,y)=0, (2)

an extension of the above said method was studied (see [8]) and is given by

$$\left.\begin{aligned} x_{n+1} &= x_n - \frac{f_x'(x_n,y_n)}{f_x''(x_n,y_n)} \\ y_{n+1} &= y_n - \frac{f_y'(x_n,y_n)}{f_y''(x_n,y_n)} \end{aligned}\right\} \tag{3}$$

4. Extension of the Method for Unconstrained Optimization Problem

Given a real valued function $f(x,y)$ defined on $X, Y \in R$. If f attains its maximum or minimum at a point $x_0 \in X$, $y_0 \in Y$, then the roots of $f_x'(x,y) = 0$ & $f_y'(x,y) = 0$ can be calculated by using Newton's interval method.

In this method when changing from fixed-point to interval based root finding, there are some immediate differences. The root is no longer a crisp interval since iterations using an interval valued function produce intervals. For details of the math and convergence properties of the algorithm, refer to Kulisch et al. (2001).

The algorithm is as follows:

$X_{k+1} = X_k \cap N_x(X_k, Y_k)$,

where

$N_x(X_k, Y_k) = m(X_k) - \frac{Fx(m(X_k), m(Y_k))}{Fx'(X_k, Y_k)}$, with $Fx'(X_k, Y_k) \neq 0$

ISSN 2229-5518

$Y_{k+1} = Y_k \cap N_y(X_k, Y_k)$,
where
$N_y(X_k, Y_k) = m(Y_k) - \frac{Fy(m(X_k), m(Y_k))}{Fy'(X_k, Y_k)}$, with $Fy'(X_k, Y_k) \neq 0$

Here X's and Y's are intervals, m(X), m(Y) are the midpoint of the interval X, Y and f is the function whose roots we seek (Kulisch et al. 35-36).

The algorithm is almost as simple as the bisection method due to an easy convergence criterion.

5. Numerical Examples

The minimum value of the function $f(x, y) = x^3 - 3xy + y^3$ can be calculated by using Interval Newton method as follows:
Here $f(x, y) = x^3 - 3xy + y^3$

$f_x'(x, y) = 3x^2 - 3y = 3(x^2 - y)$, $\quad f_y'(x, y) = -3x + 3y^2 = 3(-x + y^2)$

To find the root of $f_x'(x, y) = 3(x^2 - y)$ and $f_y'(x, y) = 3(-x + y^2)$ using Newton's Interval method.

Let $F_x = x^2 - y$ & $F_y = -x + y^2$

An interval expansion of $F_x' = 2x$ & $F_y' = 2y$ is
$F_x'(X, Y) = 2x$ & $F_y'(X, Y) = 2y$

By Newton's interval formula, we have
$X_{k+1} = X_k \cap N_x(X_k, Y_k)$,
where $N_x(X_k, Y_k) = m(X_k) - \frac{Fx(m(X_k), m(Y_k))}{Fx'(X_k, Y_k)}$

$Y_{k+1} = Y_k \cap N_y(X_k, Y_k)$,
where $N_y(X_k, Y_k) = m(Y_k) - \frac{Fy(m(X_k), m(Y_k))}{Fy'(X_k, Y_k)}$
Let us take $X_0 = [1,2]$ & $Y_0 = [1,2]$

Then
$N_x(X_0, Y_0) = m(X_0) - \frac{Fx(m(X_0), m(Y_0))}{Fx'(X_0, Y_0)}$ &
$N_y(X_0, Y_0) = m(Y_0) - \frac{Fy(m(X_0), m(Y_0))}{Fy'(X_0, Y_0)}$
$= 1.5 - \frac{0.75}{2[1,2]}$ & $= 1.5 - \frac{0.75}{2[1,2]}$
$= [1.1250, 1.3125]$ & $= [1.1250, 1.3125]$
$X_1 = [1,2] \cap [1.1250, 1.3125]$ &
$Y_1 = [1,2] \cap [1.1250, 1.3125]$
$= [1.1250, 1.3125]$ & $= [1.1250, 1.3125]$
Again $X_2 = X_1 \cap N_x(X_1, Y_1)$ & $Y_2 = Y_1 \cap N_y(X_1, Y_1)$

Now
$N_x(X_1, Y_1) = 1.2188 - \frac{0.2666}{2.2500}$ & $N_y(X_1, Y_1) = 1.2188 - \frac{0.2666}{2.2500}$
$= [1.1003, 1.1172]$ &
$= [1.1003, 1.1172]$

$X_2 = [1.1250, 1.3125] \cap [1.1003, 1.1172]$ &
$Y_2 = [1.1250, 1.3125] \cap [1.1003, 1.1172]$
$= [1.1250, 1.1172]$ & $= [1.1250, 1.1172]$
Hence the required root can be taken as the midpoint of this interval
i.e., $\frac{[1.1250+1.1172]}{2} = 1.1211$
Continuing the process until f(x, y) is minimum at x = 1 & y = 1 by elementary calculus method.

6. Conclusion

By calculating maxima and minima for linear derivatives gives approximate value. Interval Analysis method is very useful, economic and time saving. Also, the solutions obtained are better than those obtained by elementary calculus method.

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